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Representations of the generalized Lie algebra $\mathfrak{sl}(2)_q$

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Abstract. We construct finite-dimensional irreducible representations of two quantum algebras related to the generalized Lie algebra $\mathfrak{sl}(2)_q$ introduced by Lyubashenko and Sudbery. We consider separately the cases of q generic and q at roots of unity. Some of the representations have no classical analogue even for generic q. Some of the representations have no analogue to the finite-dimensional representations of the quantized enveloping $U_q(sl(2))$, while in those that do there are different matrix elements.

1. Introduction

A number of authors [1–4] have suggested definitions of 'quantum Lie algebras', the aim being to obtain structures which bear the same relation to quantized enveloping algebras as Lie algebras do to their enveloping algebras. It is of interest to determine the representations of such quantum Lie algebras, in those cases where a notion of 'representation' is defined, and compare them with the classical representation theory. For generic values of the complex deformation parameter q it is to be expected that the representations will resemble those of the classical Lie algebras which are deformed into the quantum versions, since the representation theory of a quantized enveloping algebra is essentially the same as that of the classical Lie algebra, but the details of this resemblance will help to illuminate the nature of a quantum Lie algebra. This relationship breaks down if q is a root of unity, which is of much interest in physics, and it is therefore particularly significant to determine the representations of a quantum Lie algebra in this case.

In this paper we begin such a study by constructing finite-dimensional representations of the simplest example of the generalized Lie algebras introduced in [4]. A representation of this algebra, in the sense defined in [4], is nothing but a representation of an associative algebra, the enveloping algebra of the quantum Lie algebra. This is obtained from a larger algebra with a central element by imposing a relation giving the central element as a function of Casimir-like elements. We investigate the representations also of this larger algebra, which is possibly more natural in the context of generalized Lie algebras, and find that it has additional one-dimensional representations.

The paper is organized as follows. In section 2 we introduce explicitly the two quantum algebras that we consider. In sections 3 and 4 we construct finite-dimensional representations of these algebras for generic values of q. In sections 5 and 6 we consider the case when q is a root of unity. Section 7 contains a summary of our results.

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2. The quantum Lie algebra $\mathfrak{sl}(2)_q$

The generalized Lie algebra $\mathfrak{sl}(2)_q$ was introduced in [4], cf also [5–7]. Its enveloping algebra $\mathcal{A} \equiv U(\mathfrak{sl}(2)_q)$ is defined by equation (3.5) of [4]. For the purposes of developing the representation theory it is enough to work with the algebras $\mathcal{B}, \mathcal{F}$ [4]. The algebra $\mathcal{B}$ is generated by four generator: $X_0, X_\pm, C$ with relations:

$$q^2 X_0 X_+ - X_+ X_0 = qCX_+ \tag{2.1a}$$

$$q^{-2} X_0 X_- - X_- X_0 = -q^{-1} CX_- \tag{2.1b}$$

$$X_+ X_- - X_- X_+ = (q + q^{-1})(C - \lambda X_0) X_0 \tag{2.1c}$$

$$CX_m = X_m C \qquad m = 0, \pm 1 \tag{2.1d}$$

where $\lambda \equiv q - q^{-1}$. The algebra $\mathcal{B}$ is related to the locally finite part $\mathcal{F}$ of the simply connected quantized enveloping algebra $\bar{U}_q(\mathfrak{sl}(2))$. The algebra $\mathcal{F}$ was obtained in [4] from $\mathcal{B}$ by putting C equal to a function of the second-order Casimir:

$$C_2 = (q + q^{-1}) X_0^2 + qX_- X_+ + q^{-1} X_+ X_- \tag{2.2}$$

namely,

$$C^2 = 1 + \frac{\lambda^2}{q + q^{-1}} C_2. \tag{2.3}$$

For brevity we shall call $\mathcal{F}$ the restricted algebra. The enveloping algebra $\mathcal{A}$, on the other hand, is obtained by putting $C = 1$ [4].

We shall need a triangular decomposition of $\mathcal{B}$:

$$\mathcal{B} = \mathcal{B}_+ \otimes \mathcal{B}_0 \otimes \mathcal{B}_- \tag{2.4}$$

where $\mathcal{B}_\pm$ is generated by $X_\pm$, while $\mathcal{B}_0$ is generated by X_0, C. We shall call the abelian Lie algebra $\mathcal{H}$ generated by X_0, C the Cartan subalgebra of $\mathcal{B}$. Note that $\mathcal{B}_0$ is the enveloping algebra of $\mathcal{H}$. The same decomposition is used for the algebra $\mathcal{F}$ with relation (2.3) enforced.

Further we shall analyse algebras $\mathcal{B}$ and $\mathcal{F}$ separately.

3. Highest weight representations

Highest weight modules (HWMs) of $\mathcal{B}$ are standardly determined by a highest weight vector v_0 which is annihilated by the raising generator X_+ and on which Cartan generators act by the corresponding value of the highest weight $\Lambda \in \mathcal{H}^*$:

$$\begin{aligned} &X_+ v_0 = 0 \\ &H v_0 = \Lambda(H) v_0 \qquad H \in \mathcal{H}. \end{aligned} \tag{3.1}$$

We write $M \equiv \Lambda(X_0)$, $c \equiv \Lambda(C)$.

In particular, we shall be interested in Verma modules over $\mathcal{F}$. As in the classical case a Verma module V^Λ is a HWM of weight Λ induced from a one-dimensional representation of a Borel subalgebra $\tilde{\mathcal{B}}$, e.g. $\tilde{\mathcal{B}} = \mathcal{B}_+ \otimes \mathcal{B}_0$, on the highest weight vector, e.g. v_0. As vector spaces we have:

$$V^\Lambda \cong \mathcal{B} \otimes_{\tilde{\mathcal{B}}} v_0 = \mathcal{B}_- \otimes v_0 = \text{ linear span } \{v_k \equiv X_-^k \otimes v_0 | k \in \mathbb{Z}_+\} \tag{3.2}$$

where we have identified $1_{\mathcal{B}} \otimes v_0$ with v_0.

The action of the generators of $\mathcal{B}$ on the basis of V^Λ is given as follows

$$X_+ v_k = q^{2k-2}(c - \lambda M)([2k]_q M - q[k]_q[k-1]_q c)v_{k-1} \tag{3.3a}$$

$$X_- v_k = v_{k+1} \tag{3.3b}$$

$$X_0 v_k = (q^{2k} M - q^k [k]_q c)v_k \tag{3.3c}$$

$$C v_k = c v_k \tag{3.3d}$$

where $[k]_q \equiv (q^k - q^{-k})/\lambda$. To obtain (3.3*a*), (3.3*c*) we have used the following calculations which follow from (2.1):

$$X_0 X_-^k = X_-^k (q^{2k} X_0 - q^k [k]_q C) \tag{3.4a}$$

$$[X_+, X_-^k] = X_-^{k-1} q^{2k-2}(C - \lambda X_0)([2k]_q X_0 - q[k]_q[k-1]_q C). \tag{3.4b}$$

As in the classical case the analysis of reducibility of Verma modules is an important tool in the representation theory. This analysis starts (cf [8]) with the search for singular vectors. A singular vector v_s of a Verma module V^Λ is defined as $v_s \in V^\Lambda$, $v_s \notin \mathbb{C}v_0$ and it satisfies the following properties (cf, e.g. [8]):

$$X_+ v_s = 0 \tag{3.5a}$$

$$H v_s = \Lambda'(H) v_s \qquad H \in \mathcal{H},\ \Lambda' \in \mathcal{H}^*. \tag{3.5b}$$

First we note that since C is central its value is the same as on v_0: $c' \equiv \Lambda'(C) = c$. Further, we proceed to find the possible singular vectors using the fact that they are eigenvectors of X_0. However, the eigenvectors of X_0 are $X_-^n \otimes v_0$, all with different eigenvalues. Thus, a singular vector will be given by the classical expression (omitting the overall normalization): $v_s = X_-^n \otimes v_0$ for some fixed $n \in \mathbb{N}$, and we have:

$$X_0 v_s = M' v_s \qquad M' \equiv \Lambda'(X_0) = q^{2n} M - q^n [n] q^c. \tag{3.6}$$

Finally, we have to impose (3.5*a*) for which we calculate (using (3.4*b*)):

$$X_+ v_s = X_-^{n-1} q^{2n-2}(c - \lambda M)([2n]_q M - q[n]_q[n-1]_q c) \otimes v_0. \tag{3.7}$$

For the further analysis we suppose that the deformation parameter q is not a nontrivial root of unity. Then there are two possibilities for (3.7) to be zero, and thus, we have two possibilities to fulfil (3.5*a*):

$$M = q[n]_q[n-1]_q c/[2n]_q \tag{3.8a}$$

$$c = \lambda M. \tag{3.8b}$$

We shall analyse the two possibilities in (3.8) separately since they have very different implications; moreover, they are incompatible unless $c = M = 0$ when they coincide and which we shall treat as the partial case of (3.8*b*).

The first possibility (3.8*a*) (with $c \neq 0$) corresponds to the classical relation between n and the highest weight Λ (obtained for $q, c \to 1$): $M = (n-1)/2$. Thus, if we fix $n \in \mathbb{N}$ then $v_s = X_-^n \otimes v_0$ is a singular vector when M has the value (3.8*a*). The shifted weight Λ' corresponds to another Verma module $V^{\Lambda'}$ which is the maximal invariant submodule of V^Λ. The corresponding eigenvalue of X_0 is (cf (3.6)):

$$M' = -q[n]_q[n+1]_q c/[2n]_q. \tag{3.9}$$

Note that the Verma module $V^{\Lambda'}$ does not have a singular vector. Indeed, there is no $n' \in \mathbb{N}$ such that (3.8*a*) holds for the pair (M', n') replacing (M, n). Also (3.8*b*) cannot hold for M' since $c = \lambda M'$ will contradict (3.9).

The factor-module $L_{n,c} \cong V^{\Lambda}/V^{\Lambda'}$ is irreducible and finite-dimensional of dimension n. It has a highest weight vector $|n, c\rangle$ such that:

$$\begin{aligned} &X_+|n, c\rangle = 0 \\ &H|n, c\rangle = \Lambda(H)|n, c\rangle \qquad H \in \mathcal{H} \\ &X_-^n|n, c\rangle = 0. \end{aligned} \tag{3.10}$$

Let us denote by $w_k \equiv X_-^k|n, c\rangle$, $k = 0, 1, \ldots, n-1$, the states of $L_{n,c}$. The transformation rules for w_k are:

$$X_+ w_k = q^{2k-n}[k]_q[n-k]_q \left(\frac{c[2]_q[n]_q}{[2n]_q}\right)^2 w_{k-1} \tag{3.11a}$$

$$X_- w_k = w_{k+1} \qquad k < n-1 \tag{3.11b}$$

$$X_- w_{n-1} = 0 \tag{3.11b'}$$

$$X_0 w_k = \frac{cq^k[n]_q}{[2n]_q}([n-k]_q - q^{1-n}[k+1]_q) w_k \tag{3.11c}$$

$$C w_k = c w_k. \tag{3.11d}$$

Thus, the vector w_{n-1} is the lowest weight vector of $L_{n,c}$.

Next we introduce a bilinear form in $L_{n,c}$ by the formula:

$$(w_j, w_k) \equiv \langle n, c|X_+^j X_-^k|n, c\rangle \tag{3.12}$$

where $\langle n, c|$ is such that $\langle n, c\|n, c\rangle = 1$ and:

$$\begin{aligned} &\langle n, c|X_- = 0 \\ &\langle n, c|H = \Lambda(H)\langle n, c| \qquad H \in \mathcal{H} \\ &\langle n, c|X_+^n = 0. \end{aligned} \tag{3.13}$$

Then we obtain

$$\begin{aligned} &(w_j, w_k) = \delta_{jk} q^{k(k+1-n)} \frac{[k]_q![n-1]_q!}{[n-1-k]_q!} \left(\frac{c[2]_q[n]_q}{[2n]_q}\right)^{2k} \\ &[k]_q! \equiv [k]_q[k-1]_q \ldots [1]_q \qquad [0]_q! \equiv 1. \end{aligned} \tag{3.14}$$

Clearly, (3.14) is real-valued for real q, c. Thus, for $q, c \in \mathbb{R}$ we can turn (3.12) into a scalar product and defined the norm of the basis vectors:

$$|w_k| \equiv \sqrt{(w_k, w_k)} = q^{k(k+1-n)/2} \sqrt{\frac{[k]_q![n-1]_q!}{[n-1-k]_q!}} \left(\frac{c[2]_q[n]_q}{[2n]_q}\right)^k \tag{3.15}$$

where we have chosen the root that is positive for positive c, q. We can also introduce an orthonormal basis:

$$u_k \equiv \frac{1}{|w_k|} w_k. \tag{3.16}$$

Then we have:

$$(u_j, u_k) = \delta_{jk}. \tag{3.17}$$

The transformation rules for the basis vectors u_k are:

$$X_+ u_k = q^{k-n/2} \sqrt{[k]_q[n-k]_q} \frac{c[2]_q[n]_q}{[2n]_q} u_{k-1} \tag{3.18a}$$

$$X_- u_k = q^{k+1-n/2} \sqrt{[n-1-k]_q[k+1]_q} \frac{c[2]_q[n]_q}{[2n]_q} u_{k+1} \tag{3.18b}$$

$$X_0 u_k = \frac{cq^k[n]_q}{[2n]_q}([n-k]_q - q^{1-n}[k+1]_q)u_k \tag{3.18c}$$

$$Cu_k = cu_k. \tag{3.18d}$$

The above scalar product is invariant under the real form $\mathcal{B}_r$ of $\mathcal{B}$ defined by the antilinear anti-involution:

$$\omega(X^{\pm}) = X^{\mp} \qquad \omega(X_0) = X_0 \qquad \omega(C) = C. \tag{3.19}$$

Indeed, the algebraic relations (2.1) are preserved by ω for real q. The $\mathcal{B}_r$ invariance of the scalar product means that:

$$(w_j, Xw_k) = (\omega(X)w_j, w_k) \qquad X \in \mathcal{B} \tag{3.20}$$

which is automatically satisfied by definition (3.12). (Note that (3.20) defines (,) as the Shapovalov bilinear form [9].)

Thus, for every $n \in \mathbb{N}$ we have constructed n-dimensional irreducible representations (irreps) of $\mathcal{B}$ parametrized by $c \in \mathbb{C}, c \neq 0$, with basis w_k or u_k, $(k = 0, \ldots, n-1)$. For $q, c \in \mathbb{R}$ these are irreps of the real form $\mathcal{B}_r$, which are unitary when $q, c > 0$.

The second possibility (3.8b) has no classical analogue. It implies that if c and M are related as in (3.8b) then each vector of the basis of V^{Λ} is a singular vector. Moreover, all of them have the same weight since $M' = M$, cf (3.6). This is also clear from the transformation rules (3.3) when $c = \lambda M$:

$$X_+ v_k = 0 \tag{3.21a}$$

$$X_- v_k = v_{k+1} \tag{3.21b}$$

$$X_0 v_k = M v_k \tag{3.21c}$$

$$C v_k = \lambda M v_k. \tag{3.21d}$$

Clearly, we have an infinite sequence of embedded reducible Verma modules $V_n =$ linear span$\{v_k | k \in \mathbb{Z}_+, k \geqslant n\}$ for $n \in \mathbb{Z}_+$ as $V_n \supset V_{n+1}$, the latter being the maximal invariant submodule of the former. Note that V_n is isomorphic to a submodule of all V_m with $n > m$. Furthermore, because of the coincidence of the weights these modules are also all isomorphic to each other: $V_n \cong V_m$ for all m, n. It is also clear that for every M there is only one irreducible module, namely the one-dimensional $L_M \cong V_n / V_{n+1}$, for any n. Denoting by $|M\rangle$ the only state in L_M we have for the action on it:

$$X_+ |M\rangle = 0 \tag{3.22a}$$

$$X_- |M\rangle = 0 \tag{3.22b}$$

$$X_0 |M\rangle = M|M\rangle \tag{3.22c}$$

$$C|M\rangle = \lambda M |M\rangle. \tag{3.22d}$$

Note that the above one-dimensional irrep is different from the one-dimensional $L_{1,c}$. Indeed, although the action of $X_{\pm}$ is the same, the ratio of eigenvalues of C to X_0 here is λ, while there it is $-[2]_q/q$.

4. Highest weight representations of the restricted algebra

The highest weight representations of the restricted algebra $\mathcal{F}$ are obtained from those of $\mathcal{B}$ by imposing the relation (2.3). In particular, there is the following relation between the values of the Cartan generators:

$$c^2 = 1 + \lambda^2 \left(\frac{M^2}{q^2} + c\frac{M}{q} \right). \tag{4.1}$$

This relation has to be imposed on all formulae of the previous section. There are no essential consequences of this for generic Verma modules. For the reducible Verma modules there are more interesting consequences. First we notice that the reducibility condition (3.8*b*) is incompatible with (4.1), and thus there would be no special one-dimensional irreps like L_M, cf (3.22). So it remains to consider the combination of the reducibility condition (3.8*a*) with (4.1) from which we obtain that:

$$c = \frac{\epsilon[2n]_q}{[2]_q[n]_q} \qquad M = \frac{q[n]_q[n-1]_q c}{[2n]_q} = \frac{\epsilon q[n-1]_q}{[2]_q} \qquad \epsilon = \pm 1. \tag{4.2}$$

In this case the analogue of (3.9) is:

$$M' = -\epsilon q[n+1]_q/[2]_q. \tag{4.3}$$

Let us denote the finite-dimensional representations of $\mathcal{F}$ by $\tilde{L}_{n,\epsilon}$ and the basis by $\tilde{w}_k$, $k = 0, \ldots, n-1$. The transformation rules are:

$$X_+\tilde{w}_k = q^{2k-n}[k]_q[n-k]_q\tilde{w}_{k-1} \tag{4.4a}$$
$$X_-\tilde{w}_k = \tilde{w}_{k+1} \qquad k < n-1 \tag{4.4b}$$
$$X_-\tilde{w}_{n-1} = 0 \tag{4.4b'}$$
$$X_0\tilde{w}_k = \frac{\epsilon q^k}{[2]_q}([n-k]_q - q^{1-n}[k+1]_q)\tilde{w}_k \tag{4.4c}$$
$$C\tilde{w}_k = \frac{\epsilon[2n]_q}{[2]_q[n]_q}\tilde{w}_k. \tag{4.4d}$$

Further, the analogues of (3.14) and (3.15) are:

$$(\tilde{w}_j, \tilde{w}_k) = \delta_{jk} q^{k(k+1-n)} \frac{[k]_q![n-1]_q!}{[n-1-k]_q!} \tag{4.5}$$

$$|\tilde{w}_k| \equiv \sqrt{(\tilde{w}_k, \tilde{w}_k)} = q^{k(k+1-n)/2}[k]_q!\sqrt{\frac{[k]_q![n-1]_q!}{[n-1-k]_q!}}. \tag{4.6}$$

We can also introduce an orthonormal basis:

$$\tilde{u}_k \equiv \frac{1}{|\tilde{w}_k|}\tilde{w}_k \qquad (\tilde{u}_j, \tilde{u}_k) = \delta_{jk} \tag{4.7}$$

for which the transformation rules are:

$$X_+\tilde{u}_k = q^{k-n/2}\sqrt{[k]_q[n-k]_q\tilde{u}_{k-1}} \tag{4.8a}$$
$$X_-\tilde{u}_k = q^{k+1-n/2}\sqrt{[n-1-k]_q[k+1]_q}\tilde{u}_{k+1} \tag{4.8b}$$
$$X_0\tilde{u}_k = \frac{\epsilon q^k}{[2]_q}([n-k]_q - q^{1-n}[k+1]_q)\tilde{u}_k \tag{4.8c}$$
$$C\tilde{u}_k = \frac{\epsilon[2n]_q}{[2]_q[n]_q}\tilde{u}_k. \tag{4.8d}$$

Thus, for every $n \in \mathbb{N}$ we have constructed n-dimensional irreps of $\mathcal{F}$ parametrized by $\epsilon = \pm 1$, with bases $\tilde{w}_k$ or $\tilde{u}_k (k = 0, \ldots, n-1)$.

5. Highest weight representations at roots of unity

Here we consider representations of the algebra $\mathcal{B}$ in the case when the deformation parameter is at roots of unity. More precisely, first we consider the cases when q^2 is a primitive Nth root of unity: $q = \mathrm{e}^{\pi\mathrm{i}/N}$, $N \in \mathbb{N}+1$. Then we have:

$$[x]_q = \frac{\sin \pi x/N}{\sin \pi/N}. \tag{5.1}$$

In such cases there are additional reducibility conditions coming from (3.7) besides (3.8*a*), (3.8*b*). For this we rewrite (3.8*a*) in a more general fashion:

$$M[2n]_q = q[n]_q[n-1]_q c. \tag{3.8a'}$$

We note that from (5.1) it follows that $[N]_q = [2N]_q = 0$, so (3.8*a*′) is satisfied for $n \to N$. Thus, $v_s^N = X_-^N \otimes v_0$ is a singular vector independently of the highest weight Λ. Similar to the analysis done in [10] for the quantized enveloping algebra† $U_q(sl(2))$ all $v_s^{pN} = X^{pN} \otimes v_0$ for $p \in \mathbb{N}$ are singular vectors. We denote the Verma modules they realize by $\tilde{V}_p$, $p \in \mathbb{Z}_+$, $\tilde{V}_0 \equiv V^\Lambda$. These are embedded reducible Verma modules $\tilde{V}_p \supset \tilde{V}_{p+1}$ with the same highest weight Λ. Indeed, for any $\tilde{V}_p$ using (3.6) with $n \to pN$ we have: $M' = q^{2pN}M - q^{pN}[pN]_q c = M$.

The further analysis depends on whether there are additional singular vectors besides those just displayed. There are four cases.

We start with the case when M, c do not satisfy either (3.8*a*) or (3.8*b*). We also suppose that $c \neq 0$ when N is *even*. Then there are no additional singular vectors and there is only one irreducible N-dimensional HWM $L_{\Lambda,N} \cong \tilde{V}_p/\tilde{V}_{p+1}$ (for any p), parametrized by all pairs M, c not satisfying (3.8*a*), (3.8*b*). The action of the generators of $\mathcal{B}$ on the basis of $L_{\Lambda,N}$, which we denote by $\tilde{v}_k (k = 0, \ldots, N-1)$, is given as follows

$$X_-\tilde{v}_k = q^{2k-2}(c - \lambda M)([2k]_q M - q[k]_q[k-1]_q c)\tilde{v}_{k-1} \tag{5.2a}$$
$$X_-\tilde{v}_k = \tilde{v}_{k+1} \qquad k < N-1 \tag{5.2b}$$
$$X_-\tilde{v}_{N-1} = 0 \tag{5.2b'}$$
$$X_0\tilde{v}_k = (q^{2k}M - q^k[k]_q c)\tilde{v}_k \tag{5.2c}$$
$$C\tilde{v}_k = c\tilde{v}_k. \tag{5.2d}$$

However, unlike the Drinfel'd–Jimbo case, these finite-dimensional representations are not unitarizable, which is easily seen if one considers the analogue of the bilinear form (3.12).

Next we consider the case when M, c satisfy (3.8*a*) for some $n \in \mathbb{N}, n < N$. We also suppose that $c \neq 0$ (for any N). First we note that $n < N$ is not a restriction, since then (3.8*a*) holds also for all $n + pN$, $p \in \mathbb{Z}$. Indeed, we have:

$$\begin{aligned} q[n+pN]_q[n+pN-1]_q c/[2(n+pN)]_q &= q[n]_q[n-1]_q \cos^2(\pi p)c/[2n]_q \cos(2\pi p) \\ &= q[n]_q[n-1]_q c/[2n]_q = M. \end{aligned} \tag{5.3}$$

Thus, we have another infinite series of singular vectors $v_s'^{pN} = X_-^{n+pN} \otimes v_0$ for $p \in \mathbb{Z}_+$. They realize reducible Verma modules which we denote by $\tilde{V}'_p$, $p \in \mathbb{Z}_+$ ($\tilde{V}'_0$ is the analogue of $V^{\Lambda'}$ introduced in the non-root-of-unity case, but here it is reducible). They all have the

† We recall that although the quantized enveloping algebras $U_q(\mathfrak{g})$) were introduced for arbitrary simple Lie algebras $\mathfrak{g}$ in [11, 12], the example of $U_q(\mathfrak{sl}(2))$ was introduced in [13] as an algebra and in [14] as a Hopf algebra.

same highest weight Λ' determined by M', c with M' given by (3.6). Indeed, substituting n by $n + pN$ does not change the value of M':

$$\begin{aligned} q^{2(n+pN)}M - q^{n+pN}[n+pN]_q c &= q^{2n}M - q^{n+pN}\mathrm{e}^{\pi \mathrm{i} p}[n]_q \cos(\pi p)c \\ &= q^{2n}M - q^n[n]_q c = M'. \end{aligned} \tag{5.4}$$

Of course, after substituting M with its value from (3.8a) we obtain the expression for M' in (3.9). We have the following infinite embedding chain:

$$V^\Lambda \equiv \tilde{V}_0 \supset \tilde{V}'_0 \supset \tilde{V}_1 \supset \tilde{V}'_1 \supset \cdots \tag{5.5}$$

where all embeddings are non-composite: the embeddings $\tilde{V}_p \supset \tilde{V}'_p$ are realized by singular vectors: $X_-^n \otimes v_p$, v_p being the highest weight vector of $\tilde{V}_p$, while the embeddings $\tilde{V}'_p \supset \tilde{V}_{p+1}$ are realized by singular vectors: $X_-^{N-n} \otimes v'_p$, v'_p being the highest weight vector of $\tilde{V}'_p$.

Now, factorizing each reducible Verma module by its maximal invariant submodule we obtain that for each $n \in \mathbb{N}, n < N$ there are two finite-dimensional irreps parametrized by $c \in \mathbb{C}, c \neq 0$: $L_{n,N} \cong \tilde{V}_p/\tilde{V}'_p$ (for any p) which is $(N-n)$-dimensional. However, it turns out that the irreps from one series are isomorphic to those of the other: $L'_{n,N} \cong L_{N-n,N}$. Indeed, note that the value of M' for the Verma modules $\tilde{V}'_p$ given by (3.9) should be obtained (for consistency) also from the formula for M with n substituted by $N-n$ (since this is the reducibility condition with respect to the non-composite singular vector $X_-^{N-n} \otimes v'_p$) and indeed this is the case:

$$\begin{aligned} q[N-n]_q[N-n-1]_q c/[2(N-n)]_q &= -q[n-N]_q[n+1-N]_q c/[2(n-N)]_q \\ &= -q[n]_q[n+1]_q \cos^2(\pi N)/[2n]_q \cos(2\pi N) \\ &= -q[n]_q[n+1]_q c/[2n]_q = M'. \end{aligned}$$

Furthermore, the transformation rules for $L_{n,N}$ are the same as for $L_{n,c}$, cf (3.11), while the transformation rules for $L'_{n,N}$ are obtained from (3.11) by substituting $n \to N-n$.

Thus, we are left with one series of finite-dimensional irreps $L_{n,N}$.

Next, we consider the case when M, c satisfy (3.8b) for arbitrary c. Actually, nothing is changed from the non-root-of-unity case since the relevant formulae (3.21) and (3.22) are not changed.

Finally, we consider the case when N is even and $c = 0$. Let $\tilde{N} = N/2 \in \mathbb{N}$. In these cases there are additional reducibility conditions coming from (3.8a'). Indeed, from (5.1) it follows that $[2\tilde{N}]_{\tilde{q}} = 0$ and $[\tilde{N}]_{\tilde{q}} \neq 0$. However, if $c = 0$ then (3.8a') is again satisfied. Thus, the vector $\hat{v}_s^{\tilde{N}} = X_-^{\tilde{N}} \otimes v_0$ is a singular vector independently of the value of M. Similar to the first analysis of section 5 also all $\hat{v}_s^{p\tilde{N}} = X_-^{p\tilde{N}} \otimes v_0$ for $p \in \mathbb{N}$ are singular vectors. Note that for p even these are the singular vectors that we already have: $\hat{v}_s^{p\tilde{N}} = v_s^{\tilde{p}N}$, $\tilde{p} = p/2$. We denote the Verma modules they realize by $\hat{V}_p$, $p \in \mathbb{Z}_+$, $\hat{V}_0 \equiv V^\Lambda$. These are embedded reducible Verma modules $\hat{V}_p \supset \hat{V}_{p+1}$ with the same value of M up to sign. Indeed, for any $\hat{V}_p$ using (3.6) with $n \to p\tilde{N}$ we have: $M' = q^{2p\tilde{N}}M - q^{p\tilde{N}}[p\tilde{N}]_q c = (-1)^p M$. Certainly, for even p these are Verma modules: $\hat{V}_p = V_{\tilde{p}}$.

As above the further analysis depends on whether M, c satisfy some of (3.8a), (3.8b). However, since $c = 0$ then the only additional possibility is that also $M = 0$, which is a partial case of (3.8b), which was previously considered. Thus, further, we suppose that M, c do not satisfy either of (3.8a), (3.8b) and that $M \neq 0$.

Then there are no additional singular vectors besides $\hat{v}_s^{p\tilde{N}}$. Then for each $M \neq 0$ there is only one irreducible HWM $L_{M,\tilde{N}} \cong \hat{V}_p/\hat{V}_{p+1}$ (for any p) which is $\tilde{N}$-dimensional. The action of the generators of $\mathcal{B}$ on the basis of $L_{M,\tilde{N}}$, which we denote by $\hat{v}_k (k = 0, \ldots, \tilde{N}-1)$, is given as follows

$$X_+\hat{v}_k = -q^{2k-2}\lambda[2k]_q M^2\hat{v}_{k-1} \tag{5.6a}$$

$$X_-\hat{v}_k = \hat{v}_{k+1} \qquad k < \tilde{N} - 1 \tag{5.6b}$$

$$X_-\hat{v}_{\tilde{N}-1} = 0 \tag{5.6b'}$$

$$X_0\hat{v}_k = q^{2k}M\hat{v}_k \tag{5.6c}$$

$$C\hat{v}_k = 0. \tag{5.6d}$$

Note that if $\tilde{N}$ is odd it seems that formulae (5.6) may be obtained from (5.2) for N odd and $c = 0$ by the substitution $N \to \tilde{N}$. However, this is not the same irrep since with the same replacement the parameter q there becomes $\mathrm{e}^{\pi\mathrm{i}/N} \to \mathrm{e}^{\pi\mathrm{i}/\tilde{N}}$ while the parameter q here is $\mathrm{e}^{\pi\mathrm{i}/2\tilde{N}}$.

6. Highest weight representations at roots of unit of the restricted algebra

Here we consider representations of the restricted algebra $\mathcal{F}$ in the case when the deformation parameter is at a root of unity. We start with the case: $q = \mathrm{e}^{\pi\mathrm{i}N}$, $N \in \mathbb{N}+1$, and so (3.8a') holds. The analysis is as for the algebra $\mathcal{B}$ but imposing the relation (4.1), i.e. combining the considerations of the previous two sections.

We start with the case when M, c do not satisfy (3.8a), i.e. (4.2) does not hold. We also suppose that $c \neq 0$ when N is *even*. Then there is only one irreducible N-dimensional HWM parametrized by M, c related by (4.1), which irrep we denote by $\tilde{L}_{\Lambda,N}$. For the transformation rules we can use formulae (5.2) with (4.1) imposed.

Next we consider the case when M, c satisfy (3.8c) and $c \neq 0$, Here we should be more careful so we replace n by $n + pN$ with $n < N$. Combining the reducibility condition (3.8a) with (4.1) we first obtain that:

$$c^2 = \frac{[2(n+pN)]_q^2}{[2]_q^2[n+pN]_q^2} = \frac{[2n]_q^2}{[2]_q^2[n]_q^2}. \tag{6.1}$$

Then we recover (4.2) and (4.3) for $n < N$ which means that we have the same situation as for the unrestricted algebra at roots of unity. Thus, for each $n \in \mathbb{N}$, $n < N$ and $\epsilon = \pm 1$ there is a finite-dimensional irrep: $\tilde{L}_{n,\epsilon,N}$ which is n-dimensional. The transformation rules for $\tilde{L}_{n,\epsilon,N}$ are the same as in the non-root-of-unity case, cf (4.4).

Finally, we consider the case when N is even and $c = 0$. Let $\tilde{N} = N/2 \in \mathbb{N}$. As for the unrestricted algebra there are additional reducibility conditions, i.e. again the vector $v_s^{\tilde{N}} = X_-^{\tilde{N}} \otimes v_0$ is a singular vector. However, because of (4.1) the value of M^2 is fixed:

$$M^2 = -\tilde{q}^2/\lambda^2 \qquad M = \epsilon\mathrm{i}\tilde{q}/\lambda \qquad \epsilon = \pm 1. \tag{6.2}$$

Otherwise, the analysis goes through and there is only one irreducible $\tilde{N}$-dimensional HWM $\tilde{L}_{\epsilon,\tilde{N}}$ parametrized by ϵ. The action of the generators of $\mathcal{B}$ on the basis of $\tilde{L}_{\epsilon,\tilde{N}}$, which we denote by $\hat{v}'_k (k = 0, \ldots, \tilde{N} - 1)$, is given as follows

$$X_+\hat{v}'_k = \frac{\tilde{q}^{2k}[2k]_{\tilde{q}}}{\lambda}\hat{v}'_{k-1} \tag{6.3a}$$

$$X_-\hat{v}'_k = \hat{v}'_{k+1} \qquad k < \tilde{N} - 1 \tag{6.3b}$$

$$X_-\hat{v}'_{\tilde{N}-1} = 0 \tag{6.3b'}$$

$$X_0\hat{v}'_k = \frac{\epsilon \mathrm{i}\tilde{q}^{2k+1}}{\lambda}\hat{v}'_k \tag{6.3c}$$

$$C\hat{v}'_k = 0. \tag{6.3d}$$

The crucial feature of these two irreps is that they do not have a classical limit for $\tilde{q} \to 1$ (obtained for $N \to \infty$).

7. Summary

Below by q *generic* we shall understand that q is a nonzero complex number which is not a nontrivial root of unity. We have constructed the following finite-dimensional irreps of the algebras $\mathcal{B}$ and $\mathcal{F}$.

For the algebra $\mathcal{B}$:

- $L_{n,c}$, $n \in \mathbb{N}$, $c \in \mathbb{C}$, $c \neq 0$, q generic, $\dim L_{n,c} = n$, cf (3.11), (3.18);
- L_M, $M \in \mathbb{C}$, $c = \lambda M$, q arbitrary, $\dim L_M = 1$, cf (3.22);
- $L_{\Lambda,N}$, $N \in \mathbb{N}+1$, $q = \mathrm{e}^{\pi\mathrm{i}/N}$, $M, c \in \mathbb{C}$ arbitrary not satisfying (3.8*a*), (3.8*b*), $c \neq 0$ for N even, $\dim L_{\Lambda,N} = N$, cf (5.2);
- $L_{n,c,N}$, $n, N \in \mathbb{N}$, $n < N$, $q = \mathrm{e}^{\pi\mathrm{i}/N}$, $c \in \mathbb{C}$, $c \neq 0$, $\dim L_{n,c,N} = n$, cf (3.11);
- $L_{M,\tilde{N}}$, $N = 2\tilde{N} \in 2\mathbb{N}$, $q = \mathrm{e}^{\pi\mathrm{i}/N}$, $M \in \mathbb{C}$, $M \neq 0$, $c = 0$, $\dim L_{M,\tilde{N}} = \tilde{N}$, cf (6.3).

For the algebra $\mathcal{B}_r$ with $q \in \mathbb{R}$, $q \neq 0$:

- $L_{n,c}$, $n \in \mathbb{N}$, $c \in \mathbb{R}$, $c \neq 0$, $\dim L_{n,c} = n$, cf (3.11), (3.18); unitary for $q, c > 0$.

For the algebra $\mathcal{F}$:

- $\tilde{L}_{n,\epsilon}$, $n \in \mathbb{N}$, $\epsilon = \pm 1$, q generic, $\dim L_{n,c} = n$, cf (4.4), (4.8);
- $\tilde{L}_{\Lambda,N}$, $N \in \mathbb{N}+1$, $q = \mathrm{e}^{\pi\mathrm{i}/N}$, $M, c \in \mathbb{C}$ related by (4.1) and not satisfying (3.8*a*), (3.8*b*), $c \neq 0$ for N even, $\dim L_{\Lambda,N} = N$, cf (5.2);
- $\tilde{L}_{n,\epsilon,N}$, $n, N \in \mathbb{N}$, $n < N$, $q = \mathrm{e}^{\pi\mathrm{i}/N}$, $\epsilon = \pm 1$, $\dim \tilde{L}_{n,\epsilon,N} = n$, cf (4.4);
- $\tilde{L}_{\epsilon,\tilde{N}}$, $N = 2\tilde{N} \in 2\mathbb{N}$, $q = \mathrm{e}^{\pi\mathrm{i}/N}$, $\dim \tilde{L}_{\epsilon,\tilde{N}} = \tilde{N}$, cf (6.3).

Of the above irreps only $L_{n,c}$ and $\tilde{L}_{n,\epsilon}$ have classical $\mathfrak{sl}(2)$, $\mathfrak{su}(2)$ counterparts. For fixed n for both cases this is the n-dimensional HWM of $\mathfrak{sl}(2)$ or $\mathfrak{su}(2)$ with the conjugation ω. The latter HWM is obtained from $L_{n,c}$, $\tilde{L}_{n,\epsilon}$, resp., for $q, c \to 1$, $q, \epsilon \to 1$, resp.

Of the above irreps all but L_M, $L_{M,\tilde{N}}$, $\tilde{L}_{\epsilon,\tilde{N}}$ have analogues in the representation theory [10] of the quantized enveloping algebra $U_q(\mathfrak{sl}(2))$. However, the matrix elements there are given by expressions different from ours.

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